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Lubricated plane slider bearing: solution of the inlet problem with upstream free surface

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The present work studies the problem of the flow field around the inlet section in the case of a plane slider bearing with upstream free surface. In order to determine the pressure head build-up, the inlet region of the bearing can be studied by considering a rigid plane sheet (pad) parallel to a plane wall (guide) which slides at a steady speed in a viscous incompressible fluid. The mathematical model described is based on the application of the theory of analytic functions. Through Dini's equation, which relates the real and the complex part of any analytic function, defined on the boundary of a domain, it is possible to determine the distribution of pressure on this boundary and the geometry of the free surface. In this way the problem is formally solved; however, it is difficult to obtain the solution because the thickness of the free surface is unknown. It would be possible to adopt an iterative method but the great difficulty associated with this method of solution induced the authors to follow a simplified technique. Through this study it was possible to determine the pressure build-up at the inlet; this quantity is the essential boundary condition to evaluate the actual load capacity of a slider bearing.

1. Introduction

Many studies, both theoretical and experimental, have been carried out to determine flow characteristics around the inlet of a plane slider bearing. These studies showed that the discontinuity of the film geometry at the leading edge of the pad produces a pressure build-up, whose effects are not considered by the 'elementary lubrication theory', which assumes over-pressure vanishing at both the entrance and exit of the gap.

Unlike the hypotheses of simplified models, the flow field does not present any discontinuity; this is true for slider bearings, hydrostatic journal bearings with lubricant supply pockets, etc. The moving wall drags the fluid layer but no discontinuity is present either along or across the direction of the flow; in particular, in the transition zone around the inlet section, the velocity distribution changes continuously.

In general two cases exist: in the first part (Q_1) of the flow enters the gap while another part (Q_2) flows back; secondly, when the ratio between the thickness of the unperturbed film and film thickness at the gap inlet decreases, the reflux phenomenon tends to zero ($Q_2 = 0$); if this ratio further decreases, the incident flow cannot fill the whole passage, starting at the leading edge of the pad, and the lubricated zone begins at a point further downstream (starved conditions). The present work considers only

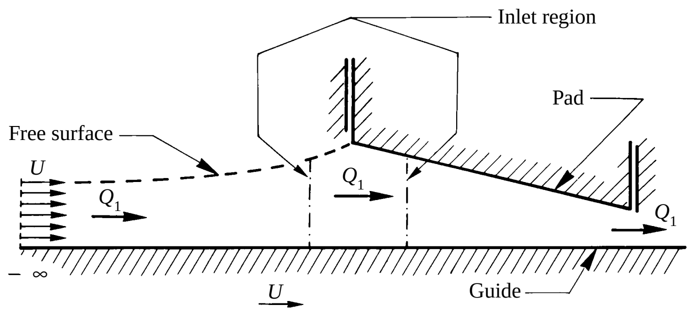

FIGURE 1. Actual bearing geometry.

the case in which $Q_2 = 0$, shown in figure 1. This case cannot be considered a general case, but it includes all the situations in which the free surface meets the pad in a section inside the channel, downstream of the inlet. The load capacity of an actual bearing depends on the location of the stagnation point; however this fact does not influence the results obtained in this work.

Several researchers have studied the problem of determining the flow field across the inlet section of a channel. Schlichting (1934) studied the development of a laminar flow between two parallel plates. He calculated the fluid velocity distribution at a number of locations downstream of the inlet section. However, in the case of a slider bearing Schlichting's model cannot apply; in fact it supposes that the inlet section does not influence the upstream flow and the transition for both velocity profiles and pressure occurs inside the channel, downstream of the inlet. Moreover in Schlichting's model inertia plays the main role inside the channel; in the case of hydrodynamic lubrication nearly the opposite occurs: in consequence of the narrowness of the film, viscous forces are dominant and inside the bearing inertia can be neglected. As a consequence the flow profile rapidly tends towards Poiseuille's profile and is completely developed in a section very close to the inlet section. On the other hand the flow is strongly disturbed upstream of the inlet section. Van Dyke (1970) and Wilson (1971) studied this model; however there was not a big difference with Schlichting's work.

The geometric discontinuity at the inlet of a slider bearing causes both the transition zone just described and the phenomenon of pressure build-up. The elementary theory of lubrication supposes that the inlet pressure p_i is the same as the ambient pressure p_e. However, in certain cases, like lubrication at high speed, the pressure head at the inlet may be not negligible, thus affecting the estimate of the performance of a lubricated slider bearing. Therefore, though on the one hand researchers agree with the existence of this physical phenomenon, proved by experimental results, on the other hand they disagree with the theoretical model that must be adopted. Some authors ascribed pressure head build-up to inertia inside the bearing; these effects have been studied both in the laminar and turbulent flow cases. A large body of literature exists that deals with the problem of turbulent hydrodynamic lubrication (Wilcock 1950; Smith & Fuller 1956; Galetuse 1974; Burton, Carper & Hsu 1974); all these studies consider cases in which the Reynolds number reaches a critical value Re_{cr}. When the flow is laminar, the fluid flow is governed by the reduced Reynolds number:

$$Re^* = Re\frac{h_o}{L} = \frac{U h_o}{\nu}\frac{h_o}{L}$$

where L is a bearing reference length (in the direction of flow), U is a reference longitudinal speed, h_o is a reference film thickness, and ν is lubricant kinematic viscosity. Lubrication theory requires that $h_o/L \ll Re < Re_{cr}$ and $Re^* \ll 1$; if $Re < Re_{cr}$ and Re^* is not much less than 1, the flow is laminar but the fluid inertia effects may be significant.

Tichy & Chen (1985) experimentally measured the load capacity in a plane slider bearing working in laminar regime at values of Re^* of order 1; the load capacity they measured was significantly higher than that determined by the elementary theory of lubrication. The theory, which they proposed to account for this discrepancy, combines an existing model, which considers an inlet pressure jump, with a model in which inertial effects inside the bearing have been studied by adopting small-perturbation analysis. Tuck & Bentwich (1983) studied the flow inside a two-dimensional lubricated slider bearing in which viscous and inertia forces are comparable by adopting a direct numerical 'shooting' method. The results show that the pressure head build-up at the inlet depends on the Reynolds number: the higher the number, the more evident is this phenomenon. It is important to emphasize that this method only gives the ratio between the pressure at any section and the pressure at the inlet; therefore the actual load capacity is only known if the true boundary conditions can be determined, namely the pressure and the velocity profile at the leading edge of the slider bearing. Therefore, it is evident that the study of the flow field around the inlet section must relate the upstream and the downstream flow.

Tipei (1978) related the pressure head at the inlet to the shape of the free surface, taking into account the surface tension in the case in which the thickness of the free surface is of the same order as the film thickness. In the Buckholz (1987) model the flow inside the bearing is governed by the one-dimensional Reynolds equation (viscous flow) while upstream of the inlet section it is governed by Bernoulli's equation (perfect fluid); these two regions are related, imposing a mass–flux balance at the leading edge. This method analytically determines the over-pressure; however the problem connected with the discontinuity of the flow profile at the inlet section still exists. Therefore it is evident that to overcome these problems the whole flow field must be studied taking into account the elliptic character of the equations.

The present work describes a mathematical model based on the theory of complex variable functions; through a conform transformation it is possible to solve problems in which free surfaces (their geometry is *a priori* unknown) and rigid walls are simultaneously present.

Helmholtz (1868) and Kirchhoff (1869) applied the theory of complex variable functions to the problem of a wake produced by a plate translating at uniform speed. Levi-Civita (1907) studied a similar problem in the case of a profile with generic shape giving a great impetus to several important applications. In this way a systematic treatment of many problems has been done: the theory of liquid jets (Cisotti 1907), the theory of the motion of a profile generating a wake in a channel that is not rectilinear (Villat 1911), flows of streams between a rigid wall and a free surface (Colonnetti 1911), merging of liquid jets (Caldonazzo 1911). A thorough examination of this mathematical treatment induced the present authors to adopt this method to determine the pressure head build-up at the inlet of a lubricated slider bearing. It is important to emphasize that the perturbation generated by the inlet section does not extend to infinity, but vanishes at a distance that is of the same order as the film thickness (Malvano & Vatta 1991); therefore in this region it is possible to apply asymptotic boundary conditions. For these reasons it is evident that the model sketched in figure 1 can be represented to a good approximation by the model of figure 2 (the pad is assumed to be like a semi-infinite wall, parallel to the guide).

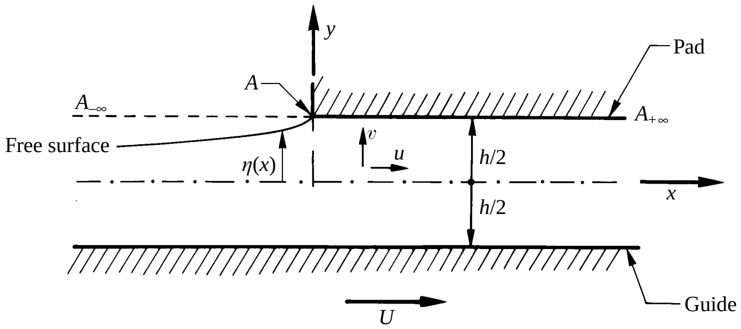

FIGURE 2. Slider bearing geometry.

Moreover in this model the boundary conditions on the free surface have been imposed on the geometric line $AA_{-\infty}$ that obviously is not a streamline. This hypothesis is justified by the fact that the actual distance between the free surface and the line $AA_{-\infty}$ is very small compared to the film thickness. Applying Dini's expression to this rectangular field, the problem, from a theoretical point of view, is formally solved.

This work describes in detail the method adopted to overcome the difficulties connected with the singularity which is present in Dini's equation. The work gives the distribution of pressure and vorticity on the boundary of the field and the geometry of the free surface.

2. Analysis

In the case of steady two dimensional incompressible laminar flow at constant viscosity and $Re^* \ll 1$, the governing equations are given by the Cauchy–Riemann differential equations:

$$\frac{\partial P}{\partial x} = \frac{\partial \Omega}{\partial y}, \quad \frac{\partial P}{\partial y} = -\frac{\partial \Omega}{\partial x} \tag{2.1}$$

where

$$P = \frac{p - p_\infty}{2\mu}, \quad \Omega = -\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right).$$

From (2.1) $\nabla^2 \Omega = 0$; introducing the stream function Ψ, we obtain

$$\frac{\partial^4 \Psi}{\partial x^4} + 2\frac{\partial^4 \Psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \Psi}{\partial y^4} = 0. \tag{2.2}$$

Therefore Ψ is a biharmonic function. Adopting the reference system z, $\bar{z}$ where

$$z = x + \mathrm{i}\,y, \quad \bar{z} = x - \mathrm{i}\,y,$$

equation (2.2) becomes

$$\frac{\partial^4 \Psi}{\partial z^2 \partial \bar{z}^2} = 0. \tag{2.3}$$

The general solution is given by

$$\Psi = \tfrac{1}{8}\mathrm{i}\,[z\,F_2(\bar{z}) - \bar{z}\,F_1(z)] + F_3(z) + F_4(\bar{z}). \tag{2.4}$$

As Ψ is a real function, as the physics of the problem requires, we have

$$F_2(\bar{z}) = \overline{F_1}(z), \quad F_4(\bar{z}) = \overline{F_3}(z) \tag{2.5}$$

where $\overline{F}$ represents the conjugate of the analytic function F. In the reference system z, $\bar{z}$ the governing equations, according to Wirtinger derivatives, are given by the following expressions:

$$\frac{\partial P}{\partial \bar{z}} = -2\,\mathrm{i}\,\frac{\partial^3 \Psi}{\partial z\,\partial \bar{z}^2}, \quad \frac{\partial P}{\partial z} = 2\,\mathrm{i}\,\frac{\partial^3 \Psi}{\partial z^2\,\partial \bar{z}}. \tag{2.6}$$

Introducing expression (2.4) into (2.6), we obtain

$$\frac{\partial P}{\partial \bar{z}} = \tfrac{1}{4}\,F_2''(\bar{z}), \quad \frac{\partial P}{\partial z} = \tfrac{1}{4}\,F_1''(z). \tag{2.7}$$

From (2.7), through (2.5), we get

$$P = \tfrac{1}{4}\,[F_1'(z) + \overline{F_1'}(z)]. \tag{2.8}$$

In the same way, for Ω we obtain

$$\Omega = \frac{1}{2}\left(\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2}\right) = 2\,\frac{\partial^2 \Psi}{\partial z\,\partial \bar{z}}.$$

Introducing expression (2.4) yields

$$\Omega = \tfrac{1}{4}\mathrm{i}\,[\overline{F_1'}(z) - F_1'(z)]. \tag{2.9}$$

Because functions P and Ω satisfy Cauchy–Riemann differential equations, it is possible to define the following analytic function:

$$W(z) = P + \mathrm{i}\,\Omega = \tfrac{1}{2}\,F_1'(z). \tag{2.10}$$

Finally, velocity components u and v and their derivatives are given by the following expressions:

$$u = \tfrac{1}{8}\,[-F_1(z) - \overline{F_1}(z) + \bar{z}\,F_1'(z) + z\,\overline{F_1'}(z)] + \mathrm{i}\,[F_3'(z) - \overline{F_3'}(z)], \tag{2.11}$$

$$v = \tfrac{1}{8}\mathrm{i}\,[F_1(z) - \overline{F_1}(z) + \bar{z}\,F_1'(z) - z\,\overline{F_1'}(z)] - F_3'(z) - \overline{F_3'}(z), \tag{2.12}$$

$$\frac{\partial u}{\partial x} = \tfrac{1}{8}\,[\bar{z}\,F_1''(z) + z\,\overline{F_1''}(z)] + \mathrm{i}\,[F_3''(z) - \overline{F_3''}(z)], \tag{2.13}$$

$$\frac{\partial u}{\partial y} = \tfrac{1}{8}\mathrm{i}\,[2\,\overline{F_1''}(z) - 2\,F_1'(z) + \bar{z}\,F_1''(z) - z\,\overline{F_1''}(z)] - \overline{F_3''}(z) - F_3''(z), \tag{2.14}$$

$$\frac{\partial v}{\partial x} = \tfrac{1}{8}\mathrm{i}\,[2\,F_1'(z) - 2\,\overline{F_1'}(z) + \bar{z}\,F_1''(z) - z\,\overline{F_1''}(z)] - F_3''(z) - \overline{F_3''}(z), \tag{2.15}$$

$$\frac{\partial v}{\partial y} = -\tfrac{1}{8}\,[\bar{z}\,F_1''(z) + z\,\overline{F_1''}(z)] + \mathrm{i}\,[\overline{F_3''}(z) - F_3''(z)]. \tag{2.16}$$

From the above expressions, the pressure and kinematic characteristics of the fluid flow can be expressed through the analytic functions given in (2.4). Therefore, in order to solve the problem, it is necessary to determine these functions; this can be done by imposing the boundary conditions.

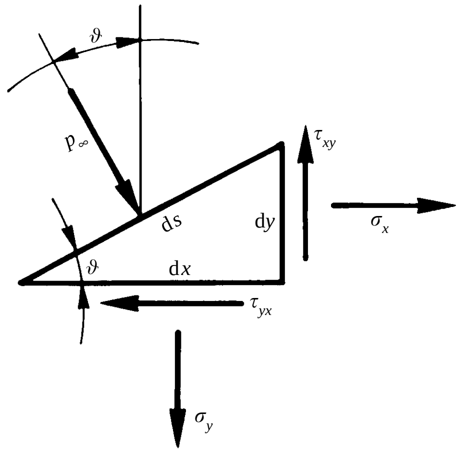

FIGURE 3. Forces acting on a fluid element of the free surface.

3. Boundary conditions

Figure 2 shows the geometry of the slider bearing and the reference system adopted.

(i) The no-slip condition on the guide gives $u + \mathrm{i}v = U$; replacing u and v with expressions (2.11) and (2.12), we obtain

$$\left\{\overline{F_3'}(z) = \mathrm{i}\,\tfrac{1}{2}U - \tfrac{1}{8}\mathrm{i}\,[z\,\overline{F_1'}(z) - F_1(z)]\right\}_{z=x-\mathrm{i}h/2}. \tag{3.1}$$

Because the relation $\mathrm{d}f/\mathrm{d}z = \partial f/\partial x$ is valid for any analytic function $f(z)$, equation (3.1) becomes

$$\overline{(F_3)}_x = -\tfrac{1}{8}x\,(F_{1i})_x - \tfrac{1}{16}h\,(F_{1r})_x - \tfrac{1}{8}\,(F_{1i}) + \mathrm{i}\left[\tfrac{1}{2}U - \tfrac{1}{8}x\,(F_{1r})_x + \tfrac{1}{16}h\,(F_{1i})_x + \tfrac{1}{8}\,(F_{1r})\right]$$

and, through (2.10), it yields on the guide

$$\overline{(F_3)}_x = -\tfrac{1}{4}x\,\Omega - \tfrac{1}{8}h\,P - \tfrac{1}{8}\,(F_{1i}) + \mathrm{i}\left[\tfrac{1}{2}U - \tfrac{1}{4}x\,P + \tfrac{1}{8}h\,\Omega + \tfrac{1}{8}\,(F_{1r})\right]. \tag{3.2}$$

The no-slip condition on the pad gives $u + \mathrm{i}v = 0$; following the same procedure as before, we get on the pad

$$\overline{(F_3)}_x = -\tfrac{1}{4}x\,\Omega + \tfrac{1}{8}h\,P - \tfrac{1}{8}\,(F_{1i}) + \mathrm{i}\left[-\tfrac{1}{4}x\,P - \tfrac{1}{8}h\,\Omega + \tfrac{1}{8}\,(F_{1r})\right]. \tag{3.3}$$

(ii) The free surface is a streamline; therefore $\Psi\,[x, \eta(x)] = \text{const}$. Introducing the coordinate $\eta(x)$ which represents the film thickness in the presence of a free surface (see figure 2), velocity components u and v on this boundary are related each other by:

$$v = u\,\frac{\mathrm{d}\eta}{\mathrm{d}x}.$$

Replacing in this relation (2.11) and (2.12), we obtain

$$(F_{3r})_x = \frac{\mathrm{d}\eta}{\mathrm{d}x}(F_{3i})_x - \tfrac{1}{8}F_{1i} + \frac{1}{8}\frac{\mathrm{d}\eta}{\mathrm{d}x}F_{1r} + \tfrac{1}{4}P\left[\eta(x) - x\,\frac{\mathrm{d}\eta}{\mathrm{d}x}\right] - \frac{\Omega}{4}\left[x + \eta(x)\,\frac{\mathrm{d}\eta}{\mathrm{d}x}\right]. \tag{3.4}$$

(iii) The equilibrium in the x-direction of a generic fluid element belonging to the free surface (figure 3) gives

$$\frac{\mathrm{d}\eta}{\mathrm{d}x}\,(\sigma_x + p_\infty) - \tau_{xy} = \frac{\mathrm{d}\eta}{\mathrm{d}x}\left(-p + 2\mu\,\frac{\partial u}{\partial x} + p_\infty\right) - \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0.$$

Through expressions (2.13), (2.14), (2.15) we obtain on the free surface:

$$(F_{3r})_{xx} = \frac{(\mathrm{d}\eta/\mathrm{d}x)\,P \,+\, \frac{1}{4}\,[\eta\,(x)\;P_x \,-\, x\,\Omega_x]\,\left[1 \,+\, \left(\mathrm{d}\eta/\mathrm{d}x\right)^2\right]}{1 \,+\, \left(\mathrm{d}\eta/\mathrm{d}x\right)^2}. \tag{3.5}$$

(iv) The equilibrium in the y-direction of a generic fluid element belonging to the free surface (figure 3) gives

$$\sigma_y \,-\, \tau_{xy}\,\frac{\mathrm{d}\eta}{\mathrm{d}x} + p_\infty \,=\, -p + 2\,\mu\,\frac{\partial v}{\partial y} \,-\, \left[\mu\left(\frac{\partial u}{\partial y} \,+\, \frac{\partial v}{\partial x}\right)\right]\frac{\mathrm{d}\eta}{\mathrm{d}x} \,+\, p_\infty = 0.$$

Through expressions (2.14), (2.15), (2.16) we obtain on the free surface

$$(F_{3i})_{xx} = \frac{\frac{1}{2}P\,\left[1 \,-\, \left(\mathrm{d}\eta/\mathrm{d}x\right)^2\right] \,+\, \frac{1}{4}\,[\eta\,(x)\;\Omega_x \,+\, x\,P_x]\,\left[1 \,+\, \left(\mathrm{d}\eta/\mathrm{d}x\right)^2\right]}{1 \,+\, \left(\mathrm{d}\eta/\mathrm{d}x\right)^2}. \tag{3.6}$$

4. Analytic solution

The relations obtained in § 3 define the analytic function $\overline{F_3''} \,=\, \overline{(F_3)}_{xx}$ on the free surface

$$\begin{aligned}\overline{(F_3)}_{xx} = {} & \frac{(\mathrm{d}\eta/\mathrm{d}x)\,P \,+\, \frac{1}{4}\,[\eta\,(x)\;P_x \,-\, x\,\Omega_x]\,\left[1 \,+\, \left(\mathrm{d}\eta/\mathrm{d}x\right)^2\right]}{1 \,+\, \left(\mathrm{d}\eta/\mathrm{d}x\right)^2} \\ & -\mathrm{i}\,\frac{\frac{1}{2}P\,\left[1 \,-\, \left(\mathrm{d}\eta/\mathrm{d}x\right)^2\right] \,+\, \frac{1}{4}\,[\eta\,(x)\;\Omega_x \,+\, x\,P_x]\,\left[1 \,+\, \left(\mathrm{d}\eta/\mathrm{d}x\right)^2\right]}{1 \,+\, \left(\mathrm{d}\eta/\mathrm{d}x\right)^2}.\end{aligned} \tag{4.1}$$

Differentiating (3.2), it is possible to define the analytic function $\overline{F_3''}$ on the guide also

$$\overline{(F_3)}_{xx} = -\tfrac{1}{4}\left[2\,\Omega + x\,\Omega_x + \tfrac{1}{2}h\,P_x\right] - \tfrac{1}{4}\mathrm{i}\,\left[x\,P_x \,-\, \tfrac{1}{2}h\,\Omega_x\right]. \tag{4.2}$$

Likewise on the pad we have:

$$\overline{(F_3)}_{xx} \,=\, -\,\tfrac{1}{4}\left[2\,\Omega \,+\, x\,\Omega_x \,-\, \tfrac{1}{2}h\,P_x\right] \,-\, \tfrac{1}{4}\mathrm{i}\,\left[x\,P_x \,+\, \tfrac{1}{2}h\,\Omega_x\right]. \tag{4.3}$$

Therefore through relations (4.1),(4.2),(4.3) it is possible to define on the whole boundary $\boldsymbol{C}$ an analytic function of the following form:

$$F\,(x) = \phi\,(x,\eta,\eta_x,\Omega,\Omega_x,P,P_x) \,+\, \mathrm{i}\,\psi\,(x,\eta,\eta_x,\Omega,\Omega_x,P,P_x)\,. \tag{4.4}$$

Pressure P and vorticity Ω are respectively the real and imaginary part of the analytic function W (see (2.10)); therefore P and Ω are not independent but are related on the boundary of the field through Dini's expression (Cisotti 1921) which is known in the case of a circle. Therefore it is necessary to find the conformal transformation to express Dini's relation in the case of the actual boundary; in general it yields

$$\Omega\,(s) = b_1 \,+\, \frac{1}{2\,\pi}\int_C \frac{\mathrm{d}P}{\mathrm{d}s_1}\,\beta\,[s\,(x)\,,s_1]\;\mathrm{d}s_1,$$

where C is the boundary of the field, s is the coordinate of a generic point on the boundary, measured starting from an arbitrary origin; $\beta\,[s\,(x)\,,s_1]$ is a function which depends on the conformal transformation; b_1 is a constant of integration.

As a consequence

$$\Omega_x = \frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_c \frac{\mathrm{d}P}{\mathrm{d}s_1}\,\beta\,(s,s_1)\,\mathrm{d}s_1\right] = \frac{1}{2\pi}\int_c \frac{\mathrm{d}P}{\mathrm{d}s_1}\,\beta_s\,(s,s_1)\;s_x\,\mathrm{d}s_1.$$

Because we expressed Ω and Ω_x versus P and its derivatives, the analytic function (4.4) becomes

$$F\,(x) = \phi\,(x,\eta,\eta_x,P,P_x) + \mathrm{i}\,\psi\,(x,\eta,\eta_x,P,P_x)\,. \tag{4.5}$$

By adopting Dini's expression once more to relate the real and the imaginary part of the analytical function F, given by (4.5), we get the integral differential equation as follows:

$$H\,(x,\eta,\eta_x,\eta_{xx},P,P_x,P_{xx}) = 0. \tag{4.6}$$

The above equation allows $P\,(x)$ to be determined; in this way the problem is solved. However the method now described supposes that the thickness $\eta\,(x)$ of the free surface is known which, in fact, is unknown. It would be possible to proceed as follows: the thickness $\eta\,(x)$ of the free surface is assumed, therefore through (4.6), $P\,(x)$ can be determined; then, through (3.4), it is possible to determine the new distribution of $\eta\,(x)$. Following this iterative method it is possible to determine the whole flow field. However, there is great difficulty in determining the conformal transformation; therefore it is necessary to adopt a simplified technique.

5. Simplified analytic solution

The flow field determination, by adopting the exact expressions obtained before, is formally solved; however the solution is difficult to obtain owing to the fact that the thickness of the free surface is unknown. Therefore Dini's expression cannot be adopted because the function which conformally transforms the circle into the actual boundary is unknown.

In order to solve the problem the dynamic equilibrium peculiar to a free surface has been imposed on $AA_{-\infty}$ (see figure 2) without imposing the condition that this line is a streamline too.

This hypothesis, that significantly simplifies the problem, is justified by the fact that the deviation of the free surface from the line $AA_{-\infty}$ is negligible in an actual case.

In conclusion, the essential assumption is that the height $h/2 - \eta\,(x)$ is small; one can thus introduce a small parameter $\varepsilon = h/2 - \eta\,(x)$. Therefore one can assume that $\overline{P}$ and $\overline{\Omega}$ (defined on the free surface) can be written as a regular perturbation expansion via $\overline{P} = P + \varepsilon P_1$, $\overline{\Omega} = \Omega + \varepsilon\Omega_1$, where P, Ω on the line $AA_{-\infty}$ are the zero-order terms.

The expressions concerning the boundary conditions on the guide and on the pad are not affected by these assumptions; on the other hand, according to the expressions adopted, the equilibrium equations on the free surface must be modified; on substituting the expansions into the governing equations, that is the boundary conditions on the free surface (3.5) and (3.6),

the equilibrium equation (3.5) along x-axis becomes

$$(F_{3r})_{xx} = \tfrac{1}{4}\left[\tfrac{1}{2}h\,P_x - x\,\Omega_x\right]; \tag{5.1}$$

the equilibrium equation (3.6) along y-axis becomes

$$(F_{3i})_{xx} = \tfrac{1}{2}P + \tfrac{1}{4}\left[x\,P_x + \tfrac{1}{2}h\,\Omega_x\right]. \tag{5.2}$$

Therefore the analytic function $\overline{F_3''}$, valid on the free surface, is given by the following expression (instead of 4.1):

$$\overline{(F_3)}_{xx} = \tfrac{1}{4}\left\{\tfrac{1}{2}h\,P_x - x\,\Omega_x\right\} - \mathrm{i}\left\{\tfrac{1}{2}P + \tfrac{1}{4}\left[x\,P_x + \tfrac{1}{2}h\,\Omega_x\right]\right\}. \tag{5.3}$$

The hypothesis adopted suggests an important conclusion: the equilibrium equation along y for an element of fluid belonging to the free surface becomes

$$p - p_\infty = 2\,\mu\,\frac{\partial v}{\partial y} = -2\,\mu\,\frac{\partial u}{\partial x},$$

that is

$$P = \frac{p - p_\infty}{2\,\mu} = -\frac{\partial u}{\partial x}.$$

Because in the neighbourhood of A $\partial u/\partial x < 0$ for $x < 0$, we have $p - p_\infty > 0$; therefore the effect of the viscosity is by itself enough to justify the presence of a pressure build-up at the inlet.

The solution can be obtained following the method shown in §4; the unknown function $\eta\,(x)$ takes the constant value $h/2$ and the boundary **C** of the field becomes a rectangle. In this case Dini's expression is given by

$$\psi\,(\vartheta) = b_0 + \frac{1}{2\,\pi}\int_0^{2\pi}\frac{\mathrm{d}\phi}{\mathrm{d}\tilde{\vartheta}}\ln\left\{4\,\sin^2\left[\frac{\vartheta - \tilde{\vartheta}}{2}\right]\right\}\,\mathrm{d}\tilde{\vartheta}, \tag{5.4}$$

where the following conform transformation must be adopted:

$$\vartheta = -\mathrm{i}\,\ln\left\{\tanh\frac{\pi\,z}{2\,h}\right\}. \tag{5.5}$$

It is useful to emphasize that the functions ϕ and ψ are respectively the real and the imaginary part of the analytic functions (4.2), (4.3), (5.3).

Through the assumptions adopted, it is possible to transform the actual flow field into a strip of thickness h. In order to obtain the numerical solution, it is necessary to determine the distance beyond which the disturbance, generated by the inlet section discontinuity, vanishes. We have showed (1991) that the distance of propagation is comparable to the film thickness. This result was obtained by using the Schwarz expression (Cisotti 1921) that relates an analytic function to its real part on the boundary. As a consequence the study of the flow field can be restricted to a finite region. This result allows the determination of the pressure build-up at the inlet and the determination of the load capacity of the whole bearing to be de-coupled: pressure build-up is obtained according to the present method and this value can be adopted as a boundary condition to determine the characteristics of a lubricated slider bearing. Therefore it is correct to find a solution in the region between two sections very close each other and symmetrically placed with respect to the inlet. In this case Dini's equation (5.4) (figure 4) becomes

$$\begin{aligned}\psi\,(\vartheta) = b_0 &+ \frac{1}{2\pi}\int_{\vartheta_0}^{\pi-\vartheta_0}\frac{\mathrm{d}\phi}{\mathrm{d}\tilde{\vartheta}}\ln\left\{4\,\sin^2\left[\frac{\vartheta - \tilde{\vartheta}}{2}\right]\right\}\mathrm{d}\tilde{\vartheta}\\ &+ \frac{1}{2\pi}\int_{\pi+\vartheta_0}^{2\pi-\vartheta_0}\frac{\mathrm{d}\phi}{\mathrm{d}\tilde{\vartheta}}\ln\left\{4\,\sin^2\left[\frac{\vartheta - \tilde{\vartheta}}{2}\right]\right\}\mathrm{d}\tilde{\vartheta}\end{aligned} \tag{5.6}$$

where the angle ϑ_0 is close to $\pi/2$; in this way it is possible to simplify (5.6).

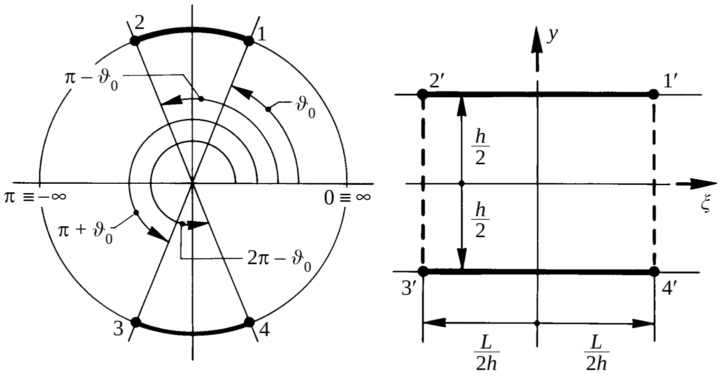

FIGURE 4. Field of integration.

If we identify the arc $\widehat{12}$ defined on the circle with the segment $\overline{1'2'}$ defined on the strip ($\widehat{12} \Rightarrow \overline{1'2'}$) Dini's expression on the free surface becomes

$$\psi\left(\xi, +\tfrac{1}{2}h\right) = b_0 + \frac{1}{2\pi}\int_{L/2h}^{0} \frac{\mathrm{d}\phi_2}{\mathrm{d}\tilde{\xi}} \ln\left(\xi - \tilde{\xi}\right)^2 \mathrm{d}\tilde{\xi}$$

$$+\frac{1}{2\pi}\int_{0}^{-L/2h} \frac{\mathrm{d}\phi_3}{\mathrm{d}\tilde{\xi}} \ln\left(\xi - \tilde{\xi}\right)^2 \mathrm{d}\tilde{\xi} + \frac{\ln 4}{2\pi}\left[\phi_1\left(+\frac{L}{2h}\right) - \phi_1\left(-\frac{L}{2h}\right)\right], \quad (5.7)$$

and on the guide

$$\psi\left(\xi, -\tfrac{1}{2}h\right) = b_0 + \frac{1}{2\pi}\int_{-L/2h}^{+L/2h} \frac{\mathrm{d}\phi_1}{\mathrm{d}\tilde{\xi}} \ln(\xi - \tilde{\xi})^2 \mathrm{d}\tilde{\xi}$$

$$+\frac{\ln 4}{2\pi}\left[\phi_2(0) - \phi_2\left(\frac{L}{2h}\right) + \phi_3\left(-\frac{L}{2h}\right) - \phi_3(0)\right], \quad (5.8)$$

where ϕ_1, ϕ_2, ϕ_3 are the real parts of the analytic function on the guide, on the pad and on the free surface respectively; ξ is the dimensionless coordinate shown in figure 4 and defined by $\xi = x/h$.

Through the expressions obtained above, pressure P can be related to the vorticity Ω on the boundary; imposing the boundary condition of no over-pressure at the ends of the field yields

$$\Omega\left(\xi, \pm\tfrac{1}{2}h\right) = b_1 \pm \frac{1}{2\pi}\int_{+L/2h}^{-L/2h} \left[\frac{\mathrm{d}P}{\mathrm{d}\tilde{\xi}}\right]_{\pm h/2} \ln(\xi - \tilde{\xi})^2 \mathrm{d}\tilde{\xi}. \quad (5.9)$$

Differentiating expression (5.9) we get

$$\Omega_\xi\left(\xi, \pm\tfrac{1}{2}h\right) = \pm\frac{1}{\pi}\int_{+L/2h}^{-L/2h} \left[\frac{\mathrm{d}P}{\mathrm{d}\tilde{\xi}}\right]_{\pm h/2} \frac{\mathrm{d}\tilde{\xi}}{\xi - \tilde{\xi}}. \quad (5.10)$$

If we assume that $\xi = (L/2h)\cos\beta$ and $\tilde{\xi} = (L/2h)\cos\tilde{\beta}$ the equation (5.10) becomes

$$\Omega_\xi\left(\beta, \pm\tfrac{1}{2}h\right) = \mp\frac{1}{\pi}\int_{0}^{\pi} P_\xi\left(\tilde{\beta}, \pm\tfrac{1}{2}h\right) \frac{\sin\tilde{\beta}}{\cos\beta - \cos\tilde{\beta}} \mathrm{d}\tilde{\beta}. \quad (5.11)$$

In the case of the pad–free surface, P_ξ is expressed with a Fourier series

$$P_\xi\left(\beta, +\tfrac{1}{2}h\right) = \sum_{m=1}^{N_u} A_m \sin(m\beta). \tag{5.12}$$

Through this assumption, which satisfies the boundary condition of zero pressure derivative at $+L/2h$ and $-L/2h$ (the distance at which the perturbation generated by the inlet section vanishes) it is possible to obtain pressure P versus β; by integrating the above expression and imposing the boundary conditions, we have

$$P(\beta) = -\frac{L}{2h}\sum_{m=2}^{N_u}\left[\frac{\sin[(m-1)\beta]}{2(m-1)} - \frac{\sin[(m+1)\beta]}{2(m+1)}\right] A_m. \tag{5.13}$$

It is worth underlining that the condition of no over-pressure for $\beta = \pi$ yields $A_1 = 0$. Substituting equation (5.12) into (5.11) yields

$$\Omega_\xi\left(\beta, +\tfrac{1}{2}h\right) = -\frac{1}{\pi}\sum_{m=2}^{N_u}\left[A_m \int_0^\pi \frac{\sin\left(m\tilde{\beta}\right)}{\cos\beta - \cos\tilde{\beta}} \sin\tilde{\beta}\,\mathrm{d}\tilde{\beta}\right].$$

The integral on the right-hand side is the well known *Glauert integral*; therefore

$$\Omega_\xi\left(\beta, +\tfrac{1}{2}h\right) = -\sum_{m=2}^{N_u} A_m \cos(m\beta). \tag{5.14}$$

Integrating equation (5.14) we obtain Ω:

$$\Omega(\beta) = -\frac{L}{2h}\sum_{m=2}^{N_u}\left[\frac{\cos[(1+m)\beta]}{2(1+m)} + \frac{\cos[(1-m)\beta]}{2(1-m)}\right] A_m + \Omega_{0u}. \tag{5.15}$$

In the case of the guide we can operate in the same way as for the pad–free surface. The Appendix gives expressions for P, Ω and for their derivatives. It is useful to emphasize that the solution of the problem has been reduced to the computation of the coefficients of the series.

In §4 and in §5 the analytic function $\overline{F}$, defined on the border of the field, has been determined. Function F, expressed in terms of ξ, is given by the following expressions:

$$F(\xi) = -\tfrac{1}{4}\left[2\Omega + \xi\Omega_\xi + \tfrac{1}{2}P_\xi\right] + \tfrac{1}{4}\mathrm{i}\left[\xi P_\xi - \tfrac{1}{2}\Omega_\xi\right] = \phi_1(\xi) + \mathrm{i}\,\psi_1(\xi), \tag{5.16}$$

$$F(\xi) = -\tfrac{1}{4}\left[2\Omega + \xi\Omega_\xi - \tfrac{1}{2}P_\xi\right] + \tfrac{1}{4}\mathrm{i}\left[\xi P_\xi + \tfrac{1}{2}\Omega_\xi\right] = \phi_2(\xi) + \mathrm{i}\,\psi_2(\xi), \tag{5.17}$$

$$F(\xi) = \tfrac{1}{4}\left[\tfrac{1}{2}P_\xi - \xi\Omega_\xi\right] + \tfrac{1}{4}\mathrm{i}\left[2P + \xi P_\xi + \tfrac{1}{2}\Omega_\xi\right] = \phi_3(\xi) + \mathrm{i}\,\psi_3(\xi). \tag{5.18}$$

Imposing Dini's expression on the function F we obtain

guide:

$$\psi_1\left(\beta, -\tfrac{1}{2}h\right) = b_0 + \int_\pi^0 \frac{\mathrm{d}\phi_1}{\mathrm{d}\tilde{\xi}} \Lambda(\beta, \tilde{\beta})\,\mathrm{d}\tilde{\beta} + \frac{\ln 4}{2\pi}\left[\phi_2\left(\tfrac{1}{2}\pi\right) - \phi_2(0) + \phi_3(\pi) - \phi_3\left(\tfrac{1}{2}\pi\right)\right], \tag{5.19}$$

pad:

$$\psi_2\left(\beta, +\tfrac{1}{2}h\right) = b_0 + \int_0^{\pi/2} \frac{d\phi_2}{d\tilde{\xi}} \Lambda(\beta, \tilde{\beta})\, d\tilde{\beta} + \int_{\pi/2}^{\pi} \frac{d\phi_3}{d\tilde{\xi}} \Lambda(\beta, \tilde{\beta})\, d\tilde{\beta} + \frac{\ln 4}{2\pi}\,[\phi_1(0) - \phi_1(\pi)], \tag{5.20}$$

free surface:

$$\psi_3\left(\beta, +\tfrac{1}{2}h\right) = b_0 + \int_0^{\pi/2} \frac{d\phi_2}{d\tilde{\xi}} \Lambda(\beta, \tilde{\beta})\, d\tilde{\beta} + \int_{\pi/2}^{\pi} \frac{d\phi_3}{d\tilde{\xi}} \Lambda(\beta, \tilde{\beta})\, d\tilde{\beta} + \frac{\ln 4}{2\pi}\,[\phi_1(0) - \phi_1(\pi)], \tag{5.21}$$

where

$$\Lambda(\beta, \tilde{\beta}) = \frac{1}{2\pi} \ln\left(\frac{L}{2h}\cos\beta - \frac{L}{2h}\cos\tilde{\beta}\right)^2 \left(-\frac{L}{2h}\sin\tilde{\beta}\right),$$

$$\frac{d\phi_1}{d\tilde{\xi}} = \sum_{m=2}^{N_d} B_m \left[-\tfrac{3}{4}\cos(m\tilde{\beta}) - \frac{m\cos\tilde{\beta}\,\sin(m\tilde{\beta})}{4\sin\tilde{\beta}} + \frac{h}{4L}\,m\,\frac{\cos(m\tilde{\beta})}{\sin\tilde{\beta}}\right],$$

$$\frac{d\phi_2}{d\tilde{\xi}} = \sum_{m=2}^{N_u} A_m \left[\tfrac{3}{4}\cos(m\tilde{\beta}) + \frac{m\cos\tilde{\beta}\,\sin(m\tilde{\beta})}{4\sin\tilde{\beta}} - \frac{h}{4L}\,m\,\frac{\cos(m\tilde{\beta})}{\sin\tilde{\beta}}\right],$$

$$\frac{d\phi_3}{d\tilde{\xi}} = \sum_{m=2}^{N_u} A_m \left[\tfrac{1}{4}\cos(m\tilde{\beta}) + \frac{m\cos\tilde{\beta}\,\sin(m\tilde{\beta})}{4\sin\tilde{\beta}} - \frac{h}{4L}\,m\,\frac{\cos(m\tilde{\beta})}{\sin\tilde{\beta}}\right].$$

The expressions (5.19), (5.20), (5.21) are the integral differential equations which give the coefficients of the series. Note that the unknown quantities are not only the coefficients A_m and B_m but also the constants of integration b_0, Ω_{0u}, Ω_{0d}. The problem is completely defined when the vorticity Ω takes known values at the ends of the field of integration; in particular Ω must vanish upstream, and downstream it must take the value corresponding to a linear velocity distribution, due to a pure drag motion. The boundary conditions on the vorticity Ω are associated with four equations; therefore, if the total number of coefficients $N_u + N_d$ is equal to K, Dini's equation must be written for $K - 3$ points of the boundary (since the first term of such summations has index two).

In conclusion it is important to emphasize that an analytycal singularity is still present in the solution; this is evident from equations (5.19), (5.20) and (5.21) in which the function Λ diverges as β tends to $\tilde{\beta}$.

Nevertheless the singularity present in the above-mentioned equations can be defined as a 'weak' one owing to its logaritmic nature. Instead the singularity present in the functions ϕ_1, ϕ_2 and ϕ_3 has been overcome by adopting the series expansions (5.12) and (5.14), which lead to the well-known Glauert integrals.

6. Determination of the free-surface profile

Along the free surface, which is a streamline, $\Psi_1 = \text{const}$. On the segment $A_{-\infty}A$ (see figure 2) the stream function Ψ_0 is not constant but is a function of ξ, that is

$\Psi_0 = \Psi_0(\xi)$. It is possible, by adopting a first-order approximation, to write

$$\Psi_0(\xi) + \frac{\partial \Psi_0}{\partial y}\,\mathrm{d}y = \Psi_1$$

and then

$$\mathrm{d}y = \frac{\Psi_1 - \Psi_0(\xi)}{\partial \Psi_0/\partial y} = \frac{\Psi_1 - \Psi_0(\xi)}{u_0(\xi)} = \tfrac{1}{2}h - \eta(\xi) \tag{6.1}$$

with the boundary condition $\Psi_0(\xi = 0) = \Psi_1$.

Through expression (6.1) it is possible to determine the thickness $\eta(\xi)$ of the free surface once the values of $u_0(\xi)$ and $\Psi_0(\xi)$ are known. At any point ξ_i the velocity u_0 can be expressed as follows:

$$u_0(\xi_i + \Delta\xi) = u_0(\xi_i) + \left(\frac{\mathrm{d}u_0}{\mathrm{d}\xi}\right)_{\xi=\xi_i}\Delta\xi$$

where

$$\frac{\mathrm{d}u_0}{\mathrm{d}\xi} = -P(\xi) - \tfrac{3}{4}\,\Omega_\xi(\xi).$$

Therefore the velocity distribution $u_0(\xi)$ is known. $\Psi_0(\xi)$ can be determined according to the following definition:

$$\Psi_0(\xi) = \int_{-\infty}^{\xi} v_0(\xi)\,\mathrm{d}\xi.$$

The velocity $v_0(\xi)$, like $u_0(\xi)$, is given by the following expression:

$$v_0(\xi_i + \Delta\xi) = v_0(\xi_i) + \left(\frac{\mathrm{d}v_0}{\mathrm{d}\xi}\right)_{\xi=\xi_i}\Delta\xi,$$

where

$$\frac{\mathrm{d}v_0}{\mathrm{d}\xi} = -\Omega(\xi).$$

Therefore the function $\Psi_0(\xi)$ is known too; finally at any point the height $\mathrm{d}y$ can be determined.

7. Numerical results and conclusions

Numerical computations were carried out for the case of a lubricated slider bearing with the following properties:

h = film thickness $= 10^{-4}$ m, $L/h = 1$, $Re = Uh/\nu = 10$, $\nu = 5 \times 10^{-5}\,\mathrm{m}^2\,\mathrm{s}^{-1}$, $\rho = 10^3$ Kg m^{-3}.

Figures (5–9) show the results obtained by adopting a series expansion of five terms both in the case of the guide ($N_d = 5$) and in the case of the pad–free surface ($N_u = 5$). It has been checked that an increase in the terms of the series does not significantly change the numerical results. Dini's integral differential equations were written for seven points, three on the guide, two on the pad and two on the free surface. Again we checked that a different choice of the points does not change the values of the quantities shown in the figures (obviously the coefficient values of the series are different). The pressure distribution (in dimensionless form) versus ξ is shown in figure 5: the continuous line shows the distribution on the free surface–pad; the dashed line shows the distribution on the guide. The most significant result is

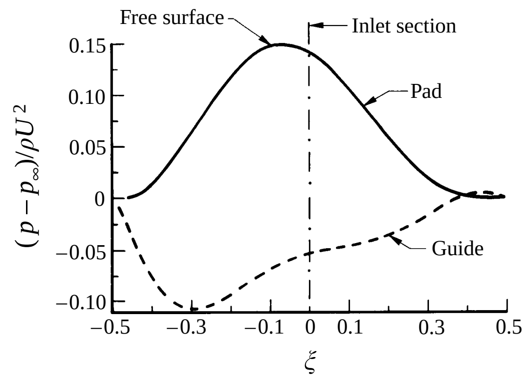

FIGURE 5. Pressure distribution along the guide and along the pad–free surface.

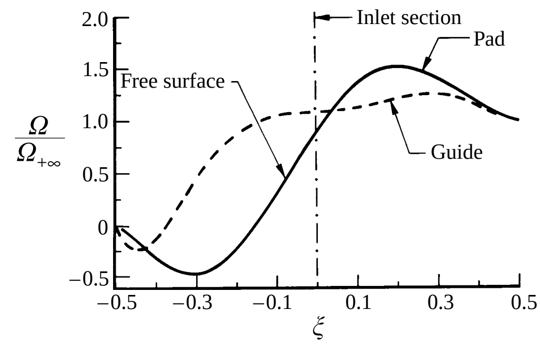

FIGURE 6. Vorticity distribution along the guide and along the pad–free surface.

given by the over-pressure which arises on the pad in the neighbourhood of the inlet section. It is worth underlining that in the present model the over-pressure is due to viscous effects only. On the other hand Tuck & Bentwich (1983) and Buckholtz (1987) refer to models in which only the inertial effects are taken into account; therefore a comparison of the results must consider these different assumptions. However it is useful to remark that the values of the over-pressure, obtained by adopting these different models, are of the same order of magnitude; therefore, in order to determine the actual boundary conditions, both the inertial effects (neglected in the present work) and the viscous effects must be considered.

On the guide all the pressure values obtained are negative. Through knowledge of the pressure distribution on the lower and upper edges of the boundary it is possible to determine the resultant force per unit width acting on the fluid element between the sections of abscissa $\xi = -0.5$ and $\xi = +0.5$. This force, perpendicular to the direction of the motion, is balanced by the resultant force due to the viscous stresses τ_{xy} produced by the downstream vorticity. The results shown in figure 5 satisfy this equilibrium condition.

Figure 6 shows the vorticity distribution versus ξ: the continuous line gives the distribution on the upper edge; the dashed line gives the distribution on the guide. The no-slip condition at the wall states that the distribution along ξ of the velocity component v is zero; therefore the vorticity Ω on the guide and on the pad is directly proportional to the viscous stress on the wall. Following the method described in § 6, the height of the free surface $\eta(\xi)$ was determined. In order to obtain this result, first

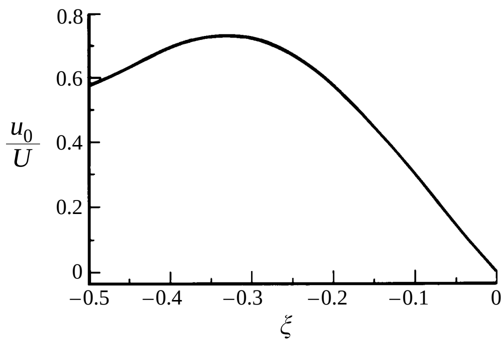

FIGURE 7. Distribution of u_0 on the line $A_{-\infty}A$.

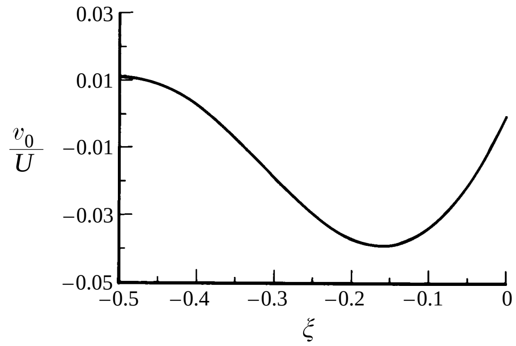

FIGURE 8. Distribution of v_0 on the line $A_{-\infty}A$.

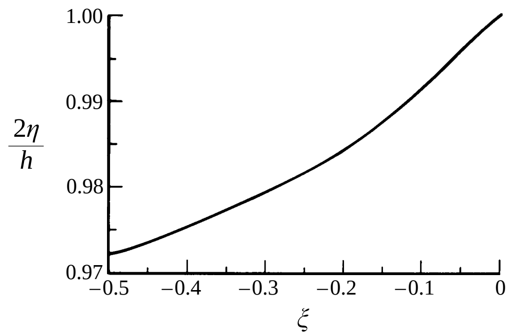

FIGURE 9. Height of the free surface in the interval $\xi = -0.5$, $\xi = 0$.

the quantities u_0 and v_0 (shown in figures 7 and 8) were determined. Through (6.1) it was possible to draw the height of the free surface, shown in figure 9 in dimensionless form. The results obtained show that the shifting of the free surface from the line $A_{-\infty}A$, in the interval $\xi = -0.5$, $\xi = 0$, is small; this result is in agreement with the hypothesis of moving the boundary conditions from the free surface onto the line $A_{-\infty}A$.

Under particular working conditions, the head build-up at the inlet is not negligible; as a consequence the actual boundary conditions must be considered for a correct determination of the bearing characteristics. However, beyond the values of the numerical results obtained, it is worth emphasizing that, through the analytical method proposed, it is possible to solve an extremely complex problem, that is the determination of a flow field whose boundary is *a priori* partially unknown.

It is important to emphasize that our solution still leads to an analytical singularity. However, through our method based on the expansion of P_ξ with a Fourier series, this singularity does not cause any problem in the numerical integration owing to the logaritmic nature of the integrand function. Following this method the solution is now obtained just through the determination of the coefficients of the series.

According to our results we can conclude that the influence of the geometrical discontinuity extends from the inlet section as far as a distance of order h. In this region the gradient of pressure and velocity is significant. The hypothesis $Q_2 = 0$ justifies the result obtained for the film thickness; it decreases from the inlet section up to upstream infinity where, due to the continuity condition, it reaches the value $h/2$. The vorticity distribution, given in figure 6, is in agreement with the above consideration. The pressure build-up at the inlet forces fluid into the bearing and causes the velocity to decrease along the free surface.

In conclusion we remark that the analytic model adopted has been used only with the aim of determining the pressure build-up. The results concerning the film thickness are only valid in the theoretical case of a bearing with parallel surfaces; on the other hand, for an actual tapered arrangement, the film thickness differs from the one mentioned above even though the results concerning the over-pressure are still valid. This last conclusion is in agreement with the results (Tichy & Chen 1985; Buckholtz 1987) which show that the pressure build-up slightly depends on the slope of the pair.

The Authors dedicate this work to Professor Carlo Ferrari and gratefully remember the several very useful discussions held with him.

Appendix.

Pad–free surface

$$P(\beta) = -\frac{L}{4h}\sum_{m=2}^{N_u}\left[\frac{\sin[(m-1)\beta]}{(m-1)} - \frac{\sin[(m+1)\beta]}{(m+1)}\right]A_m,$$

$$\Omega(\beta) = -\frac{L}{4h}\sum_{m=2}^{N_u}\left[\frac{\cos[(1+m)\beta]}{(1+m)} + \frac{\cos[(1-m)\beta]}{(1-m)}\right]A_m + \Omega_{0u},$$

$$P_\xi(\beta) = +\sum_{m=2}^{N_u} A_m \sin(m\beta), \quad \Omega_\xi(\beta) = -\sum_{m=2}^{N_u} A_m \cos(m\beta),$$

$$P_{\xi\xi}(\beta) = -\frac{2h}{L}\sum_{m=2}^{N_u} A_m\frac{\cos(m\beta)}{\sin\beta}m, \quad \Omega_{\xi\xi}(\beta) = -\frac{2h}{L}\sum_{m=2}^{N_u} A_m\frac{\sin(m\beta)}{\sin\beta}m.$$

Guide

$$P(\beta) = -\frac{L}{4h}\sum_{m=2}^{N_d}\left[\frac{\sin[(m-1)\beta]}{(m-1)} - \frac{\sin[(m+1)\beta]}{(m+1)}\right]B_m,$$

$$\Omega(\beta) = +\frac{L}{4h}\sum_{m=2}^{N_d}\left[\frac{\cos[(1+m)\beta]}{(1+m)} + \frac{\cos[(1-m)\beta]}{(1-m)}\right]B_m + \Omega_{0d},$$

$$P_{\xi}(\beta) = + \sum_{m=2}^{N_d} B_m \sin(m\beta), \quad \Omega_{\xi}(\beta) = + \sum_{m=2}^{N_d} B_m \cos(m\beta),$$

$$P_{\xi\xi}(\beta) = -\frac{2h}{L} \sum_{m=2}^{N_d} B_m \frac{\cos(m\beta)}{\sin\beta} m, \quad \Omega_{\xi\xi}(\beta) = + \frac{2h}{L} \sum_{m=2}^{N_d} B_m \frac{\sin(m\beta)}{\sin\beta} m.$$

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